AF-domains and their generalizations

Samir Bouchiba

Department of Mathematics, University Moulay Ismail, Meknes 50000, Morocco

Abstract

In this paper, we are concerned with the study of the dimension theory of tensor products of algebras over a field k. We introduce and investigate the notion of generalized AF-domain (GAF-domain for short) and prove that any k-algebra A such that the polynomial ring in one variable $A[X]$ is an AF-domain is in fact a GAF-domain, in particular any AF-domain is a GAF-domain. Moreover, we compute the Krull dimension of $A \otimes_k B$ for any k-algebra A such that $A[X]$ is an AF-domain and any k-algebra B generalizing the main theorem of Wadsworth in [16].

MSC (2000): 13C15; 13B24.

1. Introduction

All rings considered in this paper are commutative with identity element and all ring homomorphisms are unital. Throughout, k stands for a field. We shall use t.d.$(A : k)$, or t.d.(A) when no confusion is likely, to denote the transcendence degree of a k-algebra A over k (for nondomains, $\text{t.d.}(A) := \sup\left\{\text{t.d.}(\frac{A}{p}) : p \in \text{Spec}(A)\right\}$), $A[n]$ to denote the polynomial ring $A[X_1, ..., X_n]$ and $p[n]$ to denote the prime ideal $p[X_1, ..., X_n]$ of $A[X_1, ..., X_n]$ for each prime ideal p of A. Also, we use Spec(A) to denote the set of prime ideals of a ring A and $\subset$ to denote proper set inclusion. All k-algebras considered throughout this paper are assumed to be of finite transcendence degree over k. Any unreferenced material is standard as in [8], [12], [13] and [14].

Several authors have been interested in studying the prime ideal structure and related topics of tensor products of algebras over a field k. The initial impetus for these investigations was a paper of R. Sharp on Krull dimension of tensor products of two extension fields. In fact, in [15], Sharp proved that, for any two extension fields K and L of k, $\dim(K \otimes_k L) = \min(\text{t.d.}(K), \text{t.d.}(L))$ (actually, this result appeared ten years earlier in Grothendieck's EGA [10, Remarque 4.2.1.4, p. 349]). This formula is rather surprising since, as one may expect, the structure of the tensor product should reflect the way the two components interact and

E-mail address: sbouchiba@hotmail.com

keywords: Krull dimension, tensor product, prime ideal, AF-domain.

not only the structure of each component. This fact is what most motivated Wadsworth's work in [16] on this subject. His aim was to seek geometrical properties of primes of $A\otimes_k B$ and to widen the scope of algebras A and B for which $\dim(A\otimes_k B)$ depends only on individual characteristics of A and B. The algebras which proved tractable for Krull dimension computations turned out to be those domains A which satisfy the altitude formula over k (AF-domains for short), that is,

$$ht(p) + \text{t.d.}(\frac{A}{p}) = \text{ t.d.}(A)$$

for all prime ideals p of A. It is worth noting that the class of AF-domains contains the most basic rings of algebraic geometry, including finitely generated k-algebras that are domains. Wadsworth proved, via [16, Theorem 3.8], that if A_1 and A_2 are AF-domains, then

$$\dim(A_1\otimes_k A_2) = \ \min\Big(\dim(A_1) + \text{t.d.}(A_2),\ \text{t.d.}(A_1) + \dim(A_2)\Big).$$

His main theorem stated a formula for $\dim(A\otimes_k B)$ which holds for an AF-domain A, with no restriction on B, namely:

$$\begin{array}{rcl} \dim(A\otimes_k B) & = & D\Big(\text{t.d.}(A), \dim(A), B\Big) \\ & := & \max\Big\{ht(q[\text{t.d.}(A)]) + \min\Big(\text{t.d.}(A),\ \dim(A) + \text{t.d.}(\frac{B}{q})\Big) : q \in \text{Spec}(B)\Big\} \\ & & [16, \text{Theorem } 3.7]. \end{array}$$

On the other hand, in [11], Jaffard proved that, for any ring A and any positive integer n, the Krull dimension of $A[n]$ can be realized as the length of a special chain of $A[n]$. Recall that a chain $C = \{Q_0 \subset Q_1 \subset ... \subset Q_s\}$ of prime ideals of $A[n]$ is called a special chain if for each Q_i, the ideal $(Q_i \cap A)[n]$ belongs to C. Subsequently, based on the thorough and brilliant work of J. Arnold in [1], Brewer et al. gave an equivalent and simple version of Jaffard's theorem. Actually, they showed that, for each positive integer n and each prime ideal P of $A[n]$, $ht(P) = ht(q[n]) + ht(\dfrac{P}{q[n]})$ [7, Theorem 1], where $q := P \cap A$. Taking into account the natural isomorphism $B[n] \cong k[n] \otimes_k B$ for each k-algebra B, we generalized in [5] this special chain theorem to tensor products of k-algebras. Effectively, we proved that if A and B are k-algebras such that A is an AF-domain, then for each prime ideal P of $A\otimes_k B$,

$$ht(P) = ht(A\otimes_k q) + ht(\frac{P}{A\otimes_k q}) = ht(q[\text{t.d.}(A)]) + ht(\frac{P}{A\otimes_k q}),$$

where $q = P\cap B$ (cf. [5, Lemma 1.5]). It turns out that this very geometrical property totally characterizes the AF-domains. In fact, we proved, in [4], that the following statements are equivalent for a domain A which is a k-algebra:

a) A is an AF-domain;

b) A satisfies SCT (for special chain theorem), that is, for each k-algebra B and each prime ideal P of $A\otimes_k B$ with $q := P\cap B$,

$$ht(P) = ht(q[\text{t.d.}(A)]) + ht(\frac{P}{A\otimes_k q}) = ht(A\otimes_k q) + ht(\frac{P}{A\otimes_k q}) \text{ [4, Theorem 1.1]}.$$

In view of this, it is then natural to generalize the AF-domain notion by setting the following definitions:

We say that a k-algebra A satisfies GSCT (for generalized special chain theorem) with respect to a k-algebra B if

$$ht(P) = ht(p\otimes_k B + A\otimes_k q) + ht(\frac{P}{p\otimes_k B + A\otimes_k q})$$
$$\text{for each prime ideal } P \text{ of } A\otimes_k B, \text{ with } p = P\cap A \text{ and } q = P\cap B,$$

and we call a generalized AF-domain (GAF-domain for short) a domain A such that A satisfies GSCT with respect to any k-algebra B.
There is no known example in the literature of a k-algebra A which is a domain and which is not a GAF-domain. This may lead one to ask whether any k-algebra which is a domain is a GAF-domain. The object of this paper is to handle the following question:

(Q): Is any domain A which is a k-algebra such that the polynomial ring $A[n]$ is an AF-domain, for some positive integer n, a GAF-domain?

It is significant, in this regard, to note that if A is an AF-domain then $A[n]$ is an AF-domain for each integer $n\geq 0$, and using pullback constructions, Proposition 2.1 shows that these two notions do not coincide by providing a family of k-algebras A such that A is not an AF-domain while there exists a positive integer r such that the polynomial ring $A[r]$ is an AF-domain. In the present paper, we give partial results settling in the affirmative the above question (Q). First, we prove that an AF-domain A is in fact a GAF-domain, thus in particular, any finitely generated algebra over k which is a domain is a GAF-domain. Also, through Proposition 2.5, we prove that (Q) has a positive answer in the case where A is one-dimensional. Whereas, our main result, Theorem 2.8, tackles the case $n = 1$ of (Q). It computes $\dim(A\otimes_k B)$ for a k-algebra A such that $A[X]$ is an AF-domain and for an arbitrary k-algebra B generalizing Wadsworth's main theorem [16, Theorem 3.7] and further asserts that A is a GAF-domain. We end this paper by an example of a GAF-domain A such that, for any positive integer n, the polynomial ring $A[n]$ is not an AF-domain.

Recent developments on height and grade of (prime) ideals as well as on dimension theory in tensor products of k-algebras are to be found in [2-6].

2. Main results

In this section, we handle the question (Q) set above.

First, for the convenience of the reader, we catalog some basic facts and results connected with the tensor product of k-algebras. These will be used frequently in the sequel without explicit mention.

Let A and B be two k-algebras. If p is a prime ideal of A, $r = \text{t.d.}(\frac{A}{p})$ and $\overline{x_1}, ..., \overline{x_r}$ are elements of $\frac{A}{p}$, algebraically independent over k, with the $x_i \in A$, then it is easily seen that $x_1, ..., x_r$ are algebraically independent over k and $p \cap S = \emptyset$, where $S = k[x_1, ..., x_r] \setminus \{0\}$. If A is an integral domain, then $ht(p) + \text{t.d.}(\frac{A}{p}) \leq \text{t.d.}(A)$ for each prime ideal p of A (cf. [14, p. 37]). Now, assume that S_1 and S_2 are multiplicative subsets of A and B, respectively, then $S_1^{-1}A \otimes_k S_2^{-1}B \cong S^{-1}(A \otimes_k B)$, where $S = \{s_1 \otimes s_2 : s_1 \in S_1 \text{ and } s_2 \in S_2\}$. We assume familiarity with the natural isomorphisms for tensor products. In particular, we identify A and B with their respective images in $A \otimes_k B$. Also, $A \otimes_k B$ is a free (hence faithfully flat) extension of A and B. Moreover, recall that an AF-domain A is a locally Jaffard domain, that is, $ht(p[n]) = ht(p)$ for each prime ideal p and each positive integer n [16, Corollary 3.2]. Finally, we refer the reader to the useful result of Wadsworth [16, Proposition 2.3] which yields a classification of the prime ideals of $A \otimes_k B$ according to their contractions to A and B.

We begin by recalling from [2], [5] and [16] the following useful results.

Our first result allows to construct a bunch of k-algebras A arising from pullbacks which are not AF-domains while there exists an integer $n \geq 1$ such that $A[n]$ is an AF-domain.

Proposition 2.1 [5, Proposition 2.2]. *Let T be an integral domain which is a k-algebra, M a maximal ideal of T, $K := \frac{T}{M}$ and $\varphi : T \to K$ the canonical surjective homomorphism. Let D be a proper subring of K and $A := \varphi^{-1}(D)$. Assume that T and D are AF-domains. Let $r := t.d.(K : k)$ and $s = t.d.(D : k)$. Then the polynomial ring $A[r-s]$ is an AF-domain.*

Recall that, by [9], under the hypotheses of Proposition 2.1, A is an AF-domain if and only if $\text{t.d.}(K : D) = r - s = 0$. Thus, whenever $r > s$, the issued pullback A is not an AF-domain.

Proposition 2.2 [2, Lemma 1.3]. *Let A and B be k-algebras such that B is a domain. Let p be a prime ideal of A. Then, for each prime ideal P of $A \otimes_k B$ which is minimal over $p \otimes_k B$,*

$$ht(P) = ht(p \otimes_k B) = ht(p[t.d.(B)]).$$

Proposition 2.3 [16, Proposition 2.3]. *Let A and B be k-algebras and let $p \subseteq p'$ be prime ideals of A and $q \subseteq q'$ be prime ideals of B. Then the natural ring homomorphism $\varphi : \dfrac{A \otimes_k B}{p \otimes_k B + A \otimes_k q} \longrightarrow \dfrac{A}{p} \otimes_k \dfrac{B}{q}$ such that $\varphi(\overline{a \otimes_k b}) = \overline{a} \otimes_k \overline{b}$ for each $a \in A$ and each $b \in B$, is an isomorphism and*

$$\varphi(\frac{p' \otimes_k B + A \otimes_k q'}{p \otimes_k B + A \otimes_k q}) = \frac{p'}{p} \otimes_k \frac{B}{q} + \frac{A}{p} \otimes_k \frac{q'}{q}.$$

We establish the following easy result which is probably well known but we have not located references in the literature.

Proposition 2.4. *Let A be ring. Let $I \subseteq J$ be ideals in A. Then*

$$ht(I) + ht(\frac{J}{I}) \leq ht(J).$$

Proof. If both I and J are prime ideals, then the result easily follows. Fix a prime ideal Q of A that contains J. Let P be a minimal prime ideal of I contained in Q. As $ht(I) \leq ht(P)$ and $ht(P) + ht(\dfrac{Q}{P}) \leq ht(Q)$, we get $ht(I) + ht(\dfrac{Q}{P}) \leq ht(Q)$ for each minimal prime ideal P of I contained in Q. Hence $ht(I)+\max\{ht(\dfrac{Q}{P}) : P$ is a minimal prime ideal of A over I contained in $Q\} = ht(I) + ht(\dfrac{Q}{I}) \leq ht(Q)$. Therefore

$$ht(I) + ht(\frac{J}{I}) \leq ht(I) + ht(\frac{Q}{I}) \leq ht(Q)$$

for each prime ideal Q of A containing J. It follows that
$ht(I) + ht(\dfrac{J}{I}) \leq \min\{ht(Q) : J \subseteq Q \in \mathrm{Spec}(A)\} = ht(J)$, as desired. $\square$

We begin by proving that an AF-domain is a GAF-domain.

Proposition 2.5. *Let A be an AF-domain. Then A is a GAF-domain.*

Proof. Let P be a prime ideal of $A \otimes_k B$, $p = P \cap A$ and $q = P \cap B$. By the special chain theorem for tensor products [5, Lemma 1.5],

$$ht(P) = ht(A \otimes_k q) + ht(\frac{P}{A \otimes_k q}).$$

Also, note that $\dfrac{P}{A \otimes_k q}$ is a prime ideal of $A \otimes_k \dfrac{B}{q}$ such that $\dfrac{P}{A \otimes_k q} \cap \dfrac{B}{q} = (\overline{0})$. Then

$\dfrac{P}{A\otimes_k q}$ survives in the localization $A\otimes_k k_B(q)$ of $A\otimes_k \dfrac{B}{q}$, where $k_B(q)$ denotes the quotient field of $\dfrac{B}{q}$, so that, by a second application of [5, Lemma 1.5], we get

$$\begin{aligned} ht(\frac{P}{A\otimes_k q}) &= ht(p\otimes_k \frac{B}{q}) + ht\Big(\frac{P/(A\otimes_k q)}{p\otimes_k (B/q)}\Big) \\ &= ht(\frac{p\otimes_k B + A\otimes_k q}{A\otimes_k q}) + ht(\frac{P}{p\otimes_k B + A\otimes_k q}), \end{aligned}$$

as $\dfrac{p\otimes_k B + A\otimes_k q}{A\otimes_k q} \cong p\otimes_k \dfrac{B}{q}$ via Proposition 2.3. Applying Proposition 2.4, it follows that

$$\begin{aligned} ht(P) &= ht(A\otimes_k q) + ht(\frac{p\otimes_k B + A\otimes_k q}{A\otimes_k q}) + ht(\frac{P}{p\otimes_k B + A\otimes_k q}) \\ &\leq ht(p\otimes_k B + A\otimes_k q) + ht(\frac{P}{p\otimes_k B + A\otimes_k q}) \\ &\leq ht(P). \end{aligned}$$

Then the equality holds, completing the proof. $\square$

The following result settles in the affirmative the above question (Q) in the case where A is one-dimensional.

Proposition 2.6. *Let A be a one-dimensional domain such that $A[n]$ is an AF-domain for some positive integer n. Then A is a GAF-domain.*

Proof. Let B be a $k-$algebra and P a prime ideal of $A\otimes_k B$ with $p = P\cap A$ and $q = P\cap B$. Then, applying [2, Theorem 1.1], we get

$$\begin{aligned} ht(P) = \max\Big\{ & ht(q_1[\text{t.d.}(A)]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p})]) + ht(p[\text{t.d.}(\frac{B}{q_1})]) : \\ & q_1\in \text{Spec}(B) \text{ with } q_1\subseteq q\Big\} + ht(\frac{P}{p\otimes_k B + A\otimes_k q}). \end{aligned}$$

Then, for each minimal prime ideal Q of $p\otimes_k B + A\otimes_k q$, we get

$$\begin{aligned} ht(Q) = \max\Big\{ & ht(q_1[\text{t.d.}(A)]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p})]) + ht(p[\text{t.d.}(\frac{B}{q_1})]) : \\ & q_1\in \text{Spec}(B) \text{ with } q_1\subseteq q\Big\}, \end{aligned}$$

so that

$$ht(p\otimes_k B + A\otimes_k q) = \max\Big\{ht(q_1[\text{t.d.}(A)]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p})]) + ht(p[\text{t.d.}(\frac{B}{q_1})]) : q_1 \in \text{Spec}(B) \text{ with } q_1 \subseteq q\Big\}.$$

It follows that

$$ht(P) = ht(p\otimes_k B + A\otimes_k q) + ht(\frac{P}{p\otimes_k B + A\otimes_k q}).$$

Then A is a GAF-domain, as desired. $\Box$

Next, we announce the principal result of this paper. It tackles the case $n = 1$ of the above-sited question (Q) and generalizes the main theorem of Wadsworth in [16], namely: If A and B are k-algebras such that A is an AF-domain, then

$$\begin{aligned}\dim(A\otimes_k B) &= D\Big(\text{ t.d.}(A),\ \dim(A), B\Big)\\ &:= \max\Big\{ht(q[\text{t.d.}(A)]) + \min\Big(\text{t.d.}(A),\ \dim(A) + \text{t.d.}(\frac{B}{q})\Big) : q \in \text{Spec}(B)\Big\}\end{aligned}$$

[16, Theorem 3.7].

This equality might be rewritten in the following way which evokes our next general formula,

$$\begin{aligned}\dim(A\otimes_k B) &= \max\Big\{ht(q[\text{t.d.}(A)]) + ht(p[\text{t.d.}(\frac{B}{q})]) + \min\Big(\text{t.d.}(\frac{A}{p}), \text{t.d.}(\frac{B}{q})\Big) :\\ &\quad p \in \text{ Spec}(A) \text{ and } q \in \text{Spec}(B)\Big\}(\text{ as } A \text{ is a locally Jaffard domain}).\end{aligned}$$

First, it is worthwhile recalling the following definition and results from [3] and [5]. Let A and B be k-algebras and P be a prime ideal of $A\otimes_k B$. Let $q_0 \in \text{Spec}(B)$ such that $q_0 \subset P\cap B$. We denote by $\lambda\Big((., q_0), P\Big)$ the maximum of lengths of chains of prime ideals of $A\otimes_k B$ of the form $P_0 \subset P_1 \subset ... \subset P_s = P$ such that $P_i \cap B = q_0$, for $i = 0, 1, ..., s-1$. Applying [3, Lemma 2.4], if A and B are integral domains, then

$$\lambda\Big((., (0)), P\Big) \leq \text{ t.d.}(A) - \text{t.d.}(\frac{A}{p}) + ht(q[\text{t.d.}(\frac{A}{p})]) + ht(\frac{P}{p\otimes_k B + A\otimes_k q}).$$

Further, recall that, if A is a k-algebra and $n \geq 0$ is an integer, then the polynomial ring $A[n]$ is an AF-domain if and only if

$$ht(p[n]) + \text{t.d.}(\frac{A}{p}) = \text{ t.d.}(A)$$

for each prime ideal p of A [5, Lemma 2.1].

Theorem 2.7. *Let A be a k-algebra such that the polynomial ring $A[X]$ is an AF-domain. Let B be an arbitrary k-algebra. Then the following statements hold:*

a) If P is a prime ideal of $A\otimes_k B$, $p = P\cap A$ and $q = P\cap B$, then

$$ht(P) = \ max\Big\{ht(q_1[t.d.(A)]) + ht(p_1[t.d.(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[t.d.(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[t.d.(\frac{B}{q})]) :$$

$$p_1 \subseteq p \ and \ q_1 \subseteq q \ are \ prime \ ideals \ of \ A \ and \ B, \ respectively\Big\} +$$

$$ht(\frac{P}{p \otimes_k B + A \otimes_k q}).$$

b) $dim(A \otimes_k B) = \ max\Big\{ht(q_1[t.d.(A)]) + ht(p_1[t.d.(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[t.d.(\frac{A}{p_1})]) +$ $ht(\frac{p}{p_1}[t.d.(\frac{B}{q})]) + min\Big(t.d.(\frac{A}{p}), t.d.(\frac{B}{q})\Big) : p_1 \subseteq p \in Spec(A) \ and \ q_1 \subseteq q \in Spec(B)\Big\}.$

c) A is a GAF-domain.

Proof. a) **Step 1.** B is an integral domain.

If t.d.$(B) = 0$, then B is an algebraic extension field of k, and thus, by [5, Lemma 1.5], we are done. Next, assume that t.d.$(B) \geq 1$. By Proposition 2.2 and Proposition 2.3, we get, $\forall p_1 \subseteq p \in \mathrm{Spec}(A)$ and $\forall q_1 \subseteq q \in \mathrm{Spec}(B)$,

$$\begin{aligned}
ht(q_1[\text{t.d.}(A)]) &= ht(A \otimes_k q_1), \\
ht(p_1[\text{t.d.}(\frac{B}{q_1})]) &= ht(p_1 \otimes_k \frac{B}{q_1}) = ht(\frac{p_1 \otimes_k B + A \otimes_k q_1}{A \otimes_k q_1}), \\
ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) &= ht(\frac{p_1 \otimes_k B + A \otimes_k q}{p_1 \otimes_k B + A \otimes_k q_1}), \text{ and} \\
ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) &= ht(\frac{p \otimes_k B + A \otimes_k q}{p_1 \otimes_k B + A \otimes_k q}).
\end{aligned}$$

It follows that, $\forall p_1 \subseteq p \in \mathrm{Spec}(A)$ and $\forall q_1 \subseteq q \in \mathrm{Spec}(B)$,

$$ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) =$$

$$ht(A \otimes_k q_1) + ht(\frac{p_1 \otimes_k B + A \otimes_k q_1}{A \otimes_k q_1}) + ht(\frac{p_1 \otimes_k B + A \otimes_k q}{p_1 \otimes_k B + A \otimes_k q_1}) + ht(\frac{p \otimes_k B + A \otimes_k q}{p_1 \otimes_k B + A \otimes_k q}) \leq$$

$$ht(p \otimes_k B + A \otimes_k q), \text{ by Proposition 2.4.}$$

Consequently,

$$\max\Big\{ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : p_1 \subseteq p \text{ and}$$

$$q_1 \subseteq q \text{ are prime ideals of } A \text{ and } B, \text{ respectively}\Big\} + ht(\frac{P}{p \otimes_k B + A \otimes_k q}) \leq$$

$$ht(p \otimes_k B + A \otimes_k q) + ht(\frac{P}{p \otimes_k B + A \otimes_k q}) \leq ht(P).$$

Our proof of the reverse inequality uses induction on $ht(p)$ and $ht(q)$. First, note that

$$\max\Big\{ht(q[\text{t.d.}(A)]) + ht(p[\text{t.d.}(\frac{B}{q})]), ht(p[\text{t.d.}(B)]) + ht(q[\text{t.d.}(\frac{A}{p})])\Big\} \leq \tag{1}$$

$$\max\Big\{ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : p_1 \subseteq p \text{ and}$$

$$q_1 \subseteq q \text{ are prime ideals of } A \text{ and } B, \text{ respectively}\Big\}, \text{ it suffices to take}$$

$$\begin{cases} p_1 = p \\ \\ q_1 = q \end{cases} \text{and} \begin{cases} p_1 = p \\ \\ q_1 = (0). \end{cases}$$

The case where either $ht(p) = 0$ or $ht(q) = 0$ is fairly easy applying [5, Lemma 1.5]. Then, assume that $ht(p) > 0$ and $ht(q) > 0$. Suppose that $\text{t.d.}(\frac{B}{q}) \geq 1$ and let $x \in B$ such that $\overline{x}$ is a transcendental element of $\frac{B}{q}$ over k, and put $S := k[x] \setminus \{0\}$. Then,

$$\begin{cases} x \text{ is transcendental over } A \text{ (we identify } x \text{ with its image } 1 \otimes_k x \text{ through the} \\ \qquad\qquad \text{canonical injection } B \to A \otimes_k B) \\ \\ S^{-1}(A \otimes_k B) \cong A \otimes_k S^{-1}B \cong \Big(A \otimes_k k(x)\Big) \otimes_{k(x)} S^{-1}B \cong S^{-1}A[x] \otimes_{k(x)} S^{-1}B \\ \\ q \cap S = \emptyset \\ \\ S^{-1}P \in \text{Spec}\Big(S^{-1}A[x] \otimes_{k(x)} S^{-1}B\Big) \\ \\ S^{-1}A[x] \text{ is an AF-domain, by hypotheses.} \end{cases}$$

Hence

$$\begin{aligned} ht(P) &= ht(S^{-1}P) = ht\Big(S^{-1}A[x] \otimes_{k(x)} S^{-1}q\Big) + ht\Big(\frac{S^{-1}P}{S^{-1}A[x] \otimes_{k(x)} S^{-1}q}\Big), \\ &\quad \text{via [5, Lemma 1.5]} \\ &= ht\Big((A \otimes_k k(x)) \otimes_{k(x)} S^{-1}q\Big) + ht\Big(\frac{S^{-1}P}{(A \otimes_k k(x)) \otimes_{k(x)} S^{-1}q}\Big) \\ &= ht(A \otimes_k S^{-1}q) + ht(\frac{S^{-1}P}{A \otimes_k S^{-1}q}) \end{aligned}$$

$$
\begin{aligned}
&= \quad ht(A\otimes_k q)+ht(\frac{P}{A\otimes_k q})\\
&= \quad ht(q[\text{t.d.}(A)])+ht(p[\ \text{t.d.}(\frac{B}{q})])+ht(\frac{P}{p\otimes_k B+A\otimes_k q}),\ \text{by [5, Lemma 1.5]},\\
&\quad \text{since } \frac{P}{A\otimes_k q}\cap\frac{B}{q}=(0) \text{ and } p\otimes_k\frac{B}{q}\cong\frac{p\otimes_k B+A\otimes_k q}{A\otimes_k q} \text{ via Proposition 2.3,}\\
&\quad \text{so that } \frac{P/(A\otimes_k q)}{p\otimes_k(B/q)}\cong\frac{P}{p\otimes_k B+A\otimes_k q}\\
&\leq \quad \max\Big\{ht(q_1[\text{t.d.(A)}])+ht(p_1[\text{t.d.}(\frac{B}{q_1})])+ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})])+ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]):\\
&\quad p_1\subseteq p \text{ and } q_1\subseteq q \text{ are prime ideals of } A \text{ and } B,\ \text{respectively}\Big\}+\\
&\quad ht(\frac{P}{p\otimes_k B+A\otimes_k q}),\ \text{by (1)}\\
&\leq \quad ht(P),\ \text{then the equality holds, as contended.}
\end{aligned}
$$

Next, suppose that $\text{t.d.}(\frac{B}{q})=0$. Then, by [16, Proposition 2.3], P is a minimal prime ideal of $p\otimes_k B+A\otimes_k q$. Let $Q\in\text{Spec}(A\otimes_k B)$ such that $Q\subset P$ and $ht(P)=1+ht(Q)$. Let $p'=Q\cap A$ and $q'=Q\cap B$. Then either $p'\subset p$ or $q'\subset q$. Three cases arise.

Case 1. $q'\subset q$ **and** $q'\neq(0)$**.** Then $\text{t.d.}(\frac{B}{q'})\geq 1$ since $1\leq ht(\frac{q}{q'})+\text{t.d.}(\frac{B/q'}{q/q'})\leq \text{t.d.}(\frac{B}{q'})$, so that, by the above discussion,

$$ht(Q)=ht(A\otimes_k q')+ht(\frac{Q}{A\otimes_k q'}),\ \text{and hence } ht(P)=ht(A\otimes_k q')+ht(\frac{P}{A\otimes_k q'}).$$

As $q'\neq(0)$, $ht(\frac{q}{q'})<ht(q)$, then by inductive hypotheses with respect to $A\otimes_k\frac{B}{q'}$, we get

$$
\begin{aligned}
ht(\frac{P}{A\otimes_k q'}) = &\max\Big\{ht(\frac{q_1}{q'}[\text{t.d.(A)}])+ht(p_1[\text{t.d.}(\frac{B}{q_1})])+ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})])+ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]):\\
&q'\subseteq q_1\subseteq q \text{ and } p_1\subseteq p \text{ are prime ideals of } A \text{ and } B,\ \text{respectively}\Big\}+\\
&ht(\frac{P}{p\otimes_k B+A\otimes_k q}),\ \text{as, by Proposition 2.3,}
\end{aligned}
$$

$$\frac{P/(A\otimes_k q')}{p\otimes_k(B/q')+A\otimes_k(q/q')}\cong\frac{P/(A\otimes_k q')}{(p\otimes_k B+A\otimes_k q)/(A\otimes_k q')}\cong\frac{P}{p\otimes_k B+A\otimes_k q}.$$

Therefore

$$\begin{aligned}
ht(P) &= ht(q'[\text{t.d.}(A)]) + \max\Big\{ht(\frac{q_1}{q'}[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + \\
&\quad ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : p_1 \subseteq p \text{ and } q' \subseteq q_1 \subseteq q \text{ are prime ideals of} \\
&\quad A \text{ and } B, \text{ respectively}\Big\} + ht(\frac{P}{p \otimes_k B + A \otimes_k q}) \\
&= \max\Big\{ht(q'[\text{t.d.}(A)]) + ht(\frac{q_1}{q'}[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + \\
&\quad ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : p_1 \subseteq p \text{ and } q' \subseteq q_1 \subseteq q \text{ are prime ideals of} \\
&\quad A \text{ and } B, \text{ respectively}\Big\} + ht(\frac{P}{p \otimes_k B + A \otimes_k q}) \\
&\leq \max\Big\{ht(q_1[\text{t.d.}(A)]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : \\
&\quad p_1 \subseteq p \text{ and } q_1 \subseteq q \text{ are prime ideals of } A \text{ and } B, \text{ respectively}\Big\} + \\
&\quad ht(\frac{P}{p \otimes_k B + A \otimes_k q}) \\
&\leq ht(P).
\end{aligned}$$

Then the equality holds.

Case 2. $q' = q$**.** Then $p' \subset p$. By inductive hypotheses, we get

$$\begin{aligned}
ht(Q) = \ &\max\Big\{ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p'}{p_1}[\text{t.d.}(\frac{B}{q})]) : \\
&p_1 \subseteq p' \text{ and } q_1 \subseteq q \text{ are prime ideals of } A \text{ and } B, \text{ respectively}\Big\} + \\
&ht(\frac{Q}{p' \otimes_k B + A \otimes_k q}). \text{ Hence}
\end{aligned}$$

$$\begin{aligned}
ht(P) \leq \ &\max\Big\{ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p'}{p_1}[\text{t.d.}(\frac{B}{q})]) : \\
&p_1 \subseteq p' \text{ and } q_1 \subseteq q \text{ are prime ideals of } A \text{ and } B, \text{ respectively}\Big\} + \\
&ht(\frac{P}{p' \otimes_k B + A \otimes_k q}).
\end{aligned}$$

As $\dfrac{P}{p'\otimes_k B+A\otimes_k q}\cap\dfrac{B}{q}=(\overline{0})$ and $\dfrac{P}{p'\otimes_k B+A\otimes_k q}\cap\dfrac{A}{p'}=\dfrac{p}{p'}$, we get, by [5, Lemma 1.5],

$$\begin{aligned}
ht(\frac{P}{p'\otimes_k B+A\otimes_k q}) &= ht(\frac{p}{p'}[\text{t.d.}(\frac{B}{q})])+ht\Big(\frac{P/(p'\otimes_k B+A\otimes_k q)}{(p/p')\otimes_k(B/q)}\Big)\\
&= ht(\frac{p}{p'}[\text{t.d.}(\frac{B}{q})])+ht(\frac{P}{p\otimes_k B+A\otimes_k q})
\end{aligned}$$

since $\dfrac{p}{p'}\otimes_k\dfrac{B}{q}\cong\dfrac{p\otimes_k B+A\otimes_k q}{p'\otimes_k B+A\otimes_k q}$ via Proposition 2.3. It follows that

$$\begin{aligned}
ht(P) \leq\ & \max\Big\{ht(q_1[\text{t.d.(A)}])+ht(p_1[\text{t.d.}(\frac{B}{q_1})])+ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})])+ht(\frac{p'}{p_1}[\text{t.d.}(\frac{B}{q})]):\\
& p_1\subseteq p' \text{ and } q_1\subseteq q \text{ are prime ideals of } A \text{ and } B,\ \text{respectively}\Big\}+\\
& ht(\frac{p}{p'}[\text{t.d.}(\frac{B}{q})])+ht(\frac{P}{p\otimes_k B+A\otimes_k q})\\
\leq\ & \max\Big\{ht(q_1[\text{t.d.(A)}])+ht(p_1[\text{t.d.}(\frac{B}{q_1})])+ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})])+ht(\frac{p'}{p_1}[\text{t.d.}(\frac{B}{q})])+\\
& ht(\frac{p}{p'}[\text{t.d.}(\frac{B}{q})]):p_1\subseteq p' \text{ and } q_1\subseteq q \text{ are prime ideals of } A \text{ and } B,\\
& \text{respectively}\Big\}+ht(\frac{P}{p\otimes_k B+A\otimes_k q})\\
\leq\ & \max\Big\{ht(q_1[\text{t.d.(A)}])+ht(p_1[\text{t.d.}(\frac{B}{q_1})])+ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})])+ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]):\\
& p_1\subseteq p \text{ and } q_1\subseteq q \text{ are prime ideals of } A \text{ and } B,\ \text{respectively}\Big\}+\\
& ht(\frac{P}{p\otimes_k B+A\otimes_k q})\\
\leq\ & ht(P),\ \text{and then the equality holds.}
\end{aligned}$$

Case 3. $q'=(0)$**.** Then

$$\begin{aligned}
ht(P) &= 1+ht(Q)=\lambda\Big((.,(0)),P\Big)\\
&\leq \text{t.d.}(A)-\text{t.d.}(\frac{A}{p})+ht(q[\text{t.d.}(\frac{A}{p})])+ht(\frac{P}{p\otimes_k B+A\otimes_k q}),\ \text{by [3, Lemma 2.4]}\\
&= ht(p[X])+ht(q[\text{t.d.}(\frac{A}{p})])+ht(\frac{P}{p\otimes_k B+A\otimes_k q}),\ \text{as } A[X] \text{ is an AF-domain}\\
&\leq ht(p[\text{t.d.}(B)])+ht(q[\text{t.d.}(\frac{A}{p})])+ht(\frac{P}{p\otimes_k B+A\otimes_k q}),\ \text{as t.d.}(B)\geq 1
\end{aligned}$$

$$\begin{aligned}
&\leq \max\Big\{ ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : \\
&\quad p_1 \subseteq p \in \text{ Spec}(A) \text{ and } q_1 \subseteq q \in \text{ Spec}(B) \Big\} + \\
&\quad ht(\frac{P}{p \otimes_k B + A \otimes_k q}), \text{ by (1)} \\
&\leq ht(P).
\end{aligned}$$

Then the equality holds, as desired.

Step 2. B is an arbitrary k-algebra.

Let $P_0 \subset P_1 \subset ... \subset P_h = P$ be a chain of prime ideals of $A \otimes_k B$ such that $h = ht(P)$. Let $q_0 := P_0 \cap B$. Then

$$\frac{P_0}{A \otimes_k q_0} \subset \frac{P_0}{A \otimes_k q_0} \subset \frac{P_1}{A \otimes_k q_0} \subset ... \subset \frac{P_h}{A \otimes_k q_0} = \frac{P}{A \otimes_k q_0}$$

is a chain of prime ideals of $A \otimes_k \frac{B}{q_0}$ and $h = ht(P) = ht(\frac{P}{A \otimes_k q_0})$. By Step 1,

$$\begin{aligned}
ht(P) &= ht(\frac{P}{A \otimes_k q_0}) = \max\Big\{ ht(\frac{q_1}{q_0}[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + \\
&\quad ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : p_1 \subseteq p \in \text{ Spec}(A) \text{ and } q_0 \subseteq q_1 \subseteq q \in \text{ Spec}(B) \Big\} + \\
&\quad ht(\frac{P/(A \otimes_k q_0)}{p \otimes_k (B/q_0) + A \otimes_k (q/q_0)}) \\
&\leq \max\Big\{ ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + \\
&\quad ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : p_1 \subseteq p \in \text{ Spec}(A) \text{ and } q_1 \subseteq q \in \text{ Spec}(B) \Big\} + \\
&\quad ht(\frac{P}{p \otimes_k B + A \otimes_k q}) \\
&\leq ht(P), \text{ then the equality holds establishing the desired formula.}
\end{aligned}$$

b) It is a direct consequence of (a) and [16, Proposition 2.3].

c) Let B be a k-algebra. Let $p \in \text{Spec}(A)$ and $q \in \text{Spec}(B)$. Applying (a), we get

$$ht(P) = \max\Big\{ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) + ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) :$$

$$p_1 \subseteq p \text{ and } q_1 \subseteq q \text{ are prime ideals of } A \text{ and } B, \text{ respectively}\Big\}$$

for each minimal prime ideal P of $p \otimes_k B + A \otimes_k q$. It follows that

$$ht(p \otimes_k B + A \otimes_k q) = \max\Big\{ht(q_1[\text{t.d.(A)}]) + ht(p_1[\text{t.d.}(\frac{B}{q_1})]) + ht(\frac{q}{q_1}[\text{t.d.}(\frac{A}{p_1})]) +$$

$$ht(\frac{p}{p_1}[\text{t.d.}(\frac{B}{q})]) : p_1 \subseteq p \text{ and } q_1 \subseteq q \text{ are prime ideals of } A \text{ and } B, \text{ respectively}\Big\},$$

and thus

$$ht(P) = ht(p \otimes_B + A \otimes_k q) + ht(\frac{P}{p \otimes_k B + A \otimes_k q})$$

for each prime ideal P of $A \otimes_k B$ such that $p = P \cap A$ and $q = P \cap B$. Then A is a GAF-domain, as desired. $\square$

Next, we present an example of a GAF-domain A such that $A[n]$ fails to be an AF-domain for any positive integer n.

Example 2.8. Let k be an algebraically closed field. Consider the k-algebra homomorphism $\varphi : k[X, Y] \to k[[t]]$ such that $\varphi(X) = t$ and $\varphi(Y) = s := \sum_{n \geq 1} t^{n!}$. Since s is known to be transcendental over $k(t)$, φ is injective. This induces an embedding $\overline{\varphi} : k(X, Y) \to k((t))$ of fields. Put $A = \overline{\varphi}^{-1}(k[[t]])$. It is easy to check that A is a discrete rank-one valuation overring of $k[X, Y]$ of the forme $A = k + p$, where $p = XA$. Note that, for each positive integer n, and since A is Noetherian, thus a locally Jaffard domain,

$$ht(p[n]) + \text{t.d.}(\frac{A}{p}) = ht(p) + \text{t.d.}(\frac{A}{p}) = 1 < 2 = \text{ t.d.}(A).$$

Then, via [5, Lemma 2.1], for each positive integer n, $A[n]$ is not an AF-domain. Let B be an arbitrary k-algebra. Let P be a prime ideal of $A \otimes_k B$ and $q = P \cap B$. If $P \cap A = (0)$, then P survives in $k(X, Y) \otimes_k B$, and thus, by [5, Lemma 1.5], we are done. Now, assume that $P \cap A = p$. Then, as $\text{t.d.}(\frac{A}{p}) = 0$, P is minimal over $p \otimes_k B + A \otimes_k q$ [16, Proposition 2.3]. Moreover, since k is algebraically closed, $p \otimes_k B + A \otimes_k q$ is a prime ideal of $A \otimes_k B$, as $\frac{A \otimes_k B}{p \otimes_k B + A \otimes_k q} \cong \frac{A}{p} \otimes_k \frac{B}{q}$ is an integral domain, by [17, Corollary 1, p. 198]. Hence $P = p \otimes_k B + A \otimes_k q$, so that $ht(P) = ht(p \otimes_k B + A \otimes_k q)$. It follows that A is a GAF-domain, as desired. $\square$

References

[1] J.T. Arnold, *On the dimension theory of overrings of an integral domain*, Trans. Amer. Math. Soc., 138, 1969,313-326.

[2] S. Bouchiba, *On Krull dimension of tensor products of algebras arising from AF-domains*, J. Pure Appl. Algebra 203 (2005) 237-251.

[3] S. Bouchiba, *On chains of prime ideals of tensor products of algebras*, J. Pure Appl. Algebra 203 (2007) 237-251.

[4] S. Bouchiba, *AF-domains and Jaffard's special chain theorem*, The Arabian Journal for Science and Engineering, Vol. 30, n: 2A, (2005), 1-7.

[5] S. Bouchiba, F. Girolami, and S. Kabbaj, *The dimension of tensor products of k-algebras arising from pullbacks*, J. Pure Appl. Algebra 137 (1999), 125-138.

[6] S. Bouchiba and S. Kabbaj, *Tensor products of Cohen-Macaulay rings: Solution to a problem of Grothendieck*, J. Algebra 252 (2002) 65-73.

[7] J.W. Brewer, P.R. Montgomery, E.A. Rutter, W.J. Heinzer, *Krull dimension of polynomial rings*, Lecture Notes in Math., vol. 311, Springer-Verlag, Berlin, NY, 1972, 26-45.

[8] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[9] F. Girolami, *AF-Rings and Locally Jaffard Rings*, Lecture Notes Pure Appl. Math., Vol. 153. New York: Dekker, 1994, 151161.

[10] A. Grothendieck, *Eléments de géométrie algébrique*, Vol. IV, Institut des Hautes Etudes Sci. Publ. Math. No. 24, Bures-sur-yvette, 1965.

[11] P. Jaffard, *Théorie de la dimension dans les anneaux de polynômes*, Mém. Sc. Math. 146, Gauthier-Villars, Paris, 1960.

[12] I. Kaplansky, *Commutative rings*, Chicago univ., Chicago press, 1974.

[13] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, 1986.

[14] M. Nagata, *Local rings*, Interscience, New York, 1962.

[15] R.Y. Sharp, *The dimension of the tensor product of two field extensions*, Bull. London Math. Soc. 9 (1977), 42-48.

[16] A.R. Wadsworth, *The Krull dimension of tensor products of commutative algebras over a field*, J. London Math. Soc. 19 (1979), 391-401.

[17] O. Zariski and P. Samuel, *Commutative Algebra Vol. I*, Van Nostrand, Princeton, 1960.